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A note on the super Krichever map

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Abstract

We consider the geometrical aspects of the Krichever map in the context of Jacobian super KP hierarchy. We use the representation of the hierarchy based on the Faà di Bruno recursion relations, considered as the cocycle condition for the natural double complex associated with the deformations of super Krichever data. Our approach is based on the construction of the universal super divisor (of degree g), and a local universal family of geometric data which give the map into the Super Grassmannian. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper, we study the geometrical setting of the super Krichever map in analogy to the standard non-graded case [5]. This map is an essential tool in the analysis of the algebraic geometric solutions to integrable systems of soliton type (see, e.g., [10]). Its super extension has already been introduced in [11], and studied in [1,12]. The essential difference in our approach is that we have taken full benefit of the so-called Faà di Bruno approach to the KP theory [4] and its super generalization [6], where the equivalence of this approach to the standard differential-operator picture of the Jacobian SKP [9,12] is proved. It turns out, as in the classical case, that the Faà di Bruno recursion relation is (related to) the first cocycle condition for the hypercohomology group which controls the infinitesimal deformations of the spectral super line bundle together with its meromorphic sections.

The basic advantage of this approach is that it is directly related to the (Super)Grassmannian description of the hierarchy, and has an intrinsic geometrical meaning. In particular,

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we can avoid the difficult initial step of the introduction of the Baker–Akhiezer function, go on with the natural development of the geometrical construction, and recover the existence of the BA function at the end.

As in the classical case, the important technical tool is to construct a local universal deformation of the initial super line bundle, since the cocycle condition comes by considering point-wise the germ of such a deformation as a vector field on the base. This is a difficult point because we lack a sound definition of the Super Jacobian of a super curve $\mathcal{C}$. Indeed looking at the transition functions, one would say that this is the cohomology group $H^1(\mathcal{C}, \mathcal{O}_0^\times)$, where $\mathcal{O}_0^\times$ is the sheaf of units in the even part of the structure sheaf. Unfortunately, this set-up is not fully satisfactory because there are no naturally defined odd deformation directions. The way out we present in this paper is to work with the moduli space $S_g\tilde{\mathcal{C}}$ of effective superdivisors of degree g or, which is the same, with the g-fold symmetric product [3] of the dual curve [1,2]. This is a supervariety with enough odd parameters over which we have a universal effective divisor, and we expect that, as in the classical case, the "Super Jacobian" will appear as a quotient of $S_g\tilde{\mathcal{C}}$.

The scheme of the paper is as follows: in Section 2, we briefly recall the Faà di Bruno recursion relations, their connection with the JSKP hierarchy, and with the Krichever map, referring to [6] for more details. In Section 3, we give a brief *resumè* of the tools from deformation theory needed in the sequel. In Section 4, we construct the symmetric powers of the (dual) supercurve as a supervariety, and we prove the existence of a universal superdivisor. In Section 5, we exploit the cohomological meaning of the Faà di Bruno recursion relations to insure that it gives a flow on the space of super Krichever data and, through the super Krichever map, the JSKP flow on the algebraic geometrical loci in the Super Grassmannian. Finally, in Appendix A, we recall some basic definitions of the theory of super curves used in the paper.

2. The Jacobian super KP hierarchy

Let us start by fixing some notations. We denote by Λ a generic Grassmann algebra over $\mathbb{C}$, $B := \Lambda[[x, \varphi]]$ is the Λ-algebra of formal power series in the variables x (even) and φ (odd) and $\mathcal{D} := \partial_\varphi + \varphi\partial_x$. The ring of formal super pseudo-differential operators over $X := \mathrm{Spec}(B)$ is the space of formal series

$$L := \sum_{j\geq 0} u_j \mathcal{D}^{n-j}, \quad u_j \in B$$

endowed with the product induced by the super Leibniz rule

$$\mathcal{D}^k \cdot f = \sum_{j\geq 0} (-1)^{\bar{f}(k-j)} \begin{bmatrix} k \\ j \end{bmatrix} f^{(j)} \mathcal{D}^{k-j},$$

where $\bar{f}$ denotes the parity of f, $f^{(j)} = \mathcal{D}^j(f)$ and $\begin{bmatrix} k \\ j \end{bmatrix}$ is the super binomial coefficient [8].

Mulase and Rabin [9,12] defined the Jacobian super KP hierarchy as the following set of evolutionary equations for the even dressing operator $S := 1 + \sum_{j>0} s_j \mathcal{D}^{-j}$:

$$\partial_{t_{2k}} S := -(S\mathcal{D}^{2k}S^{-1})_- S = -(S\partial_x^k S^{-1})_- S,$$
$$\partial_{t_{2k-1}} S := -(S(\mathcal{D}^{2k-1} - \varphi\mathcal{D}^{2k})S^{-1})_- S = -(S\partial_\varphi \partial_x^{k-1} S^{-1})_- S,$$

where L_- is the pure pseudo-differential part of L and the time t_k has parity $k \bmod 2$. One of the features which distinguishes this hierarchy from that of Manin and Radul [8] is that for algebraic geometric solutions the equations describe super-commuting linear flows on the super Jacobian of a super curve. One way to approach this issue is to consider another description of the hierarchy, using the super Faà di Bruno polynomials instead of super pseudo-differential operators. We refer to [6] for a detailed account and we only sketch what is relevant to the present discussion. Let $V := \Lambda((z^{-1})) \oplus \Lambda((z^{-1})) \cdot \theta$ be the algebra of formal Laurent series in the even variable z^{-1} and the odd variable θ, let $V_+ := \Lambda[z, \theta]$, $V_- := \Lambda[[z^{-1}, \theta]] \cdot z^{-1}$ and let $V_B := V \otimes_\Lambda B$. The basic object of this formulation is the odd Faà di Bruno generator

$$\hat{h}(z, \theta; x, \varphi) := \theta + \varphi z + \mathrm{O}(z^{-1}) \in V_B,$$

where, abusing notations, we write $\mathrm{O}(z^{-1})$ for an element of $V_- \otimes_\Lambda B$. Out of $\hat{h}$ we construct iteratively the Faà di Bruno polynomials by

$$\hat{h}^{(0)} := 1, \qquad \hat{h}^{(k+1)} := (\mathcal{D} + \hat{h})\hat{h}^{(k)}, \quad k \in \mathbb{N}, \tag{2.1}$$

and set $W_B := \mathrm{span}_B\{\hat{h}^{(k)} : k \in \mathbb{N}\}$. It is then easy to show that there exists a unique basis $\{\hat{H}^{(k)}, k \in \mathbb{N}\}$ of W_B, whose elements (called “super currents”) have the form

$$\hat{H}^{(2k+p)} = \theta^p z^k + \mathrm{O}(z^{-1}) \tag{2.2}$$

with $p = 0, 1$, in terms of which the equations of the Jacobian super KP hierarchy become

$$\frac{\partial \hat{h}}{\partial t_k} = (-1)^k \mathcal{D}\hat{H}^{(k)}. \tag{2.3}$$

Since $\hat{H}^{(2)} = \hat{h}^{(2)}$, we have $\partial_{t_2} \equiv \partial_x$.

The study of these equations finds its most appropriate and natural setting in the concept of super universal Grassmannian SGr_Λ defined as follows [1,13]. The filtration $\cdots \subset V_{j-1} \subset V_j \subset V_{j+1} \subset \cdots \subset V$, where $V_j = z^{j+1}V_-$, makes V and its Λ-submodule V_+ complete topological spaces and $\mathrm{SGr}_\Lambda := \mathrm{SGr}_\Lambda(V, V_+)$ is the set of closed free Λ-submodules W of V which are compatible with V_+ in the sense that the restriction π_W of the projection $\pi : V \to V_+$ to W is a Fredholm operator, i.e., its kernel (respectively, cokernel) is a Λ-submodule (respectively, a quotient Λ-module) of a finite rank free Λ-module. As in the commutative setting, SGr_Λ is the disjoint union of the denumerable set of its components $\mathrm{SGr}_\Lambda^{(i)}$ labelled by the index i_W of π_W; moreover each component acquires a structure of super scheme by means of projective limits. By definition, the space W_B spanned by the $\hat{H}^{(k)}$’s gives rise to a moving point of SGr_Λ and the super currents evolve under JSKP along the equations of a dynamical system, the *super central system* [6], which gives vector fields on the Grassmannian. In particular one has that

$$\partial_{t_j}\hat{H}^{(k)} = (-1)^{jk}\partial_{t_k}\hat{H}^{(j)}. \tag{2.4}$$

Whichever approach one takes, the link with algebraic geometric solutions is provided by the super Krichever map [11] which associates a point W of SGr_Λ to the datum $(\mathcal{C}, D, (z^{-1}, \theta), \mathcal{L}, \eta)$ of

1. a Λ-super-curve $\mathcal{C} = (C, \mathcal{O}_{\mathcal{C}})$ (see Appendix A),
2. an irreducible divisor D on $\mathcal{C}$ whose reduced support is a smooth point $p_\infty \in C$,
3. local coordinates z^{-1} and θ in a neighbourhood $U_0 \ni p_\infty$,
4. an invertible sheaf $\mathcal{L}$ on $\mathcal{C}$ and
5. a local trivialization η of $\mathcal{L}$ over U_0.

Let $\mathcal{L}(\infty D) = \lim_{n\to\infty}\mathcal{L}(nD)$ be the sheaf of sections of $\mathcal{L}$ with at most an arbitrary pole at D, then $W = \eta(H^0(\mathcal{C}, \mathcal{L}(\infty D)))$. Bergvelt and Rabin [1] have shown that the Λ-module $H^0(\mathcal{C}, \mathcal{L}(\infty D))$ is free, so W belongs indeed to SGr_Λ. We can invert the Krichever map on its image as explained in [11]; in particular, the ring of functions of $\mathcal{C}$ which are holomorphic on the open subset $U_1 := C - \{p_\infty\}$ is the subalgebra $\mathcal{A}_W \subset V$ of functions f such that $f \cdot W \subset W$. $\mathcal{A}_W$ is obviously graded.

As a consequence of Eqs. (2.1) and (2.3), we recover the same picture in our approach.

Proposition 2.1 (Isospectrality). *Let $\hat{h}(x, \varphi, \mathbf{t})$ be a solution of the Jacobian super KP hierarchy and denote by W_T, the space generated by the corresponding super currents $\hat{H}^{(k)}(x, \varphi, \mathbf{t})$. For any specialization $(x_0, \varphi_0, \mathbf{t}_0)$ of $(x, \varphi, \mathbf{t})$ let $\mathcal{A}_{(x_0,\varphi_0,\mathbf{t}_0)} \subset V$ be the Λ-algebra of functions that map by multiplication W_{T_0} into itself. Then $\mathcal{A}_{(x_0,\varphi_0,\mathbf{t}_0)}$ does not depend on $(x_0, \varphi_0, \mathbf{t}_0)$.*

We limit ourselves to sketch the proof. We have to show that if $f \in \mathcal{A}_{(x_0,\varphi_0,\mathbf{t}_0)}$, then $fW_T \subset W_T$. Since $1 \in W_T$, this is equivalent to showing that such an f is in W_T, because f supercommutes with $\mathcal{D} + \hat{h}$. Since $1 \in W_{T_0}$ as well, we can write

$$f = \sum c_j \widetilde{H^{(j)}},$$

where $\widetilde{H^{(j)}}$ denote the specialization of $\hat{H}^{(j)}$ at $\mathbf{t} = \mathbf{t}_0$. We have to prove that, calling

$$f' = \sum c_j \hat{H}^{(j)},$$

actually $f' = f$, i.e., $\sum c_j \hat{H}^{(j)}$ is independent of the times t_k.

The identity

$$f \cdot (\mathcal{D} + \hat{h})^k = \sum_{j\geq 0}(-1)^{(j(j+1)/2)+k\bar{f}}\begin{bmatrix} k \\ j \end{bmatrix}(\mathcal{D} + \hat{h})^{k-j} f^{(j)}$$

shows that $\mathcal{D}^k f' \in W_T \forall k$ so that $\mathcal{D} f' = 0$. Similarly, one proves that $\partial_{t_k} f' = 0\,\forall k$.

3. Deformation of super line bundles and of their sections

The meaning of *isospectrality* (Proposition 2.1) is that when the solution is of algebraic geometric type the super curve $\mathcal{C}$ (also called the *spectral curve*) remains unaffected by the flows of the hierarchy. Indeed, it is also true that the divisor D and the coordinates (z^{-1}, θ) do not change, so the motion involves only the line bundle $\mathcal{L}$ and its local trivialization η. Since our aim is to geometrically interpret Eqs. (2.1) and (2.3), which are of differential type, and the super Krichever map is defined in terms of sections of a super line bundle $\mathcal{L}$, we have to understand how the sections change when we deform $\mathcal{L}$.

Definition 3.1. Let $\mathcal{C}$ be a Λ-super-curve, $\mathcal{L}$ an invertible sheaf on $\mathcal{C}$, s a global section of $\mathcal{L}$ and $(\mathcal{X}, x)$ a pointed Λ-super-scheme. An $\mathcal{X}$-family of invertible sheaves on $\mathcal{C}$ is an invertible sheaf $\mathcal{L}_{\mathcal{X}}$ over $\mathcal{C} \times_{\mathrm{Spec}(\Lambda)} \mathcal{X}$. A deformation of $(\mathcal{L}, s)$ over the pointed super-scheme $(\mathcal{X}, x)$ is a triple $(\mathcal{L}_{\mathcal{X}}, \sigma, \rho)$, where
1. $\mathcal{L}_{\mathcal{X}}$ is an $\mathcal{X}$-family of invertible sheaves on $\mathcal{C}$,
2. σ is a global section of $\mathcal{L}_{\mathcal{X}}$, and
3. ρ is an isomorphism, $\rho : \mathcal{L} \to \iota^*\mathcal{L}_{\mathcal{X}}$, where $\iota : \mathcal{C} \hookrightarrow \mathcal{C} \times_{\mathrm{Spec}(\Lambda)} \mathcal{X}$ is the embedding identifying $\mathcal{C}$ with $\mathcal{C} \times_{\mathrm{Spec}(\Lambda)} \{x\}$, such that $\iota^*\sigma = \rho s$.

Two deformations $(\mathcal{L}_{\mathcal{X}}, \sigma, \rho)$ and $(\mathcal{N}_{\mathcal{X}}, \tau, \xi)$ of $(\mathcal{L}, s)$ over $(\mathcal{X}, x)$ are isomorphic if and only if there exists an isomorphism of sheaves $\eta : \mathcal{L}_{\mathcal{X}} \to \mathcal{N}_{\mathcal{X}}$ compatible with ρ and ξ $(\xi = \iota^*(\eta) \circ \rho)$ and such that $\tau = \eta(\sigma)$. The line bundle $\mathcal{L}_{\mathcal{X}}|_{\mathcal{C}\times_{\mathrm{Spec}(\Lambda)}\{x\}} \simeq \mathcal{L}$ is sometimes called the *central fibre* of the deformation. Finally, an *infinitesimal deformation of* $(\mathcal{L}, s)$ is a deformation over the "one-point" Λ-super-scheme

$$\mathcal{E} := \mathrm{Spec}\left(\frac{\Lambda[t, \varepsilon]}{\langle t^2, t\varepsilon\rangle}\right),$$

where t is even and ε is odd.

Let $\{U_j\}_{j\in J}$ be a covering by open affine sub–super-schemes of $\mathcal{C}$ and denote by $U_{j_1,\dots,j_k}$ the intersection $\cap_{l=1}^{k} U_{j_l}$, by $\mathcal{O}_{j_1,\dots,j_k}$ the super-commutative ring of sections of $\mathcal{O}_{\mathcal{C}}$ over $U_{j_1,\dots,j_k}$, and by $\mathcal{L}_{j_1,\dots,j_k}$ the $\mathcal{O}_{j_1,\dots,j_k}$-module of sections of $\mathcal{L}$ over $U_{j_1,\dots,j_k}$. Finally, define

$$\mathcal{O}_{j_1,\dots,j_k}[t, \varepsilon] := \mathcal{O}_{j_1,\dots,j_k} \otimes_{\Lambda} \mathcal{O}_{\mathcal{E}}, \qquad \mathcal{L}_{j_1,\dots,j_k}[t, \varepsilon] := \mathcal{L}_{j_1,\dots,j_k} \otimes_{\Lambda} \mathcal{O}_{\mathcal{E}},$$
$$U_{j_1,\dots,j_k}[t, \varepsilon] := \mathrm{Spec}(\mathcal{O}_{j_1,\dots,j_k}[t, \varepsilon]) = U_{j_1,\dots,j_k} \times_{\mathrm{Spec}(\Lambda)} \mathcal{E}.$$

Then, $\{U_j[t, \varepsilon]\}_{j\in J}$ is an open affine covering of $\mathcal{C} \times_{\mathrm{Spec}(\Lambda)} \mathcal{E}$ and the exact sequence of sheaves

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}_j & \to & \mathcal{O}_j[t, \varepsilon]_0^{\times} & \to & \mathcal{O}_{j,0}^{\times} & \to & 1, \\ & & f & \mapsto & 1 + tf_0 + \varepsilon f_1, & & & & \end{array}$$

where f_0 and f_1 are the even and odd components of f, yields a group isomorphism $\mathrm{Pic}(U_j[t, \varepsilon]) \simeq \mathrm{Pic}(U_j)$ due to the fact that the U_j's are Stein (see [14], 1.3.8). Thus, if $\mathcal{L}_{\mathcal{E}}$ is an infinitesimal deformation of $\mathcal{L}$ then

$$\mathcal{L}_{\mathcal{E}}|_{U_j[t,\varepsilon]} \simeq (\mathcal{L}|_{U_j})[t, \varepsilon],$$

so it is described as the gluing of the last modules by means of a suitable isomorphism

$$G_{jk} : \mathcal{L}_{jk}[t,\varepsilon] \xrightarrow{\simeq} \mathcal{L}_{jk}[t,\varepsilon],$$

which in turn is given by the transition matrix

$$G_{jk} = \begin{pmatrix} g_{jk} & 0 & 0 \\ \delta_t g_{jk} & g_{jk} & 0 \\ \delta_\varepsilon g_{jk} & 0 & g_{jk} \end{pmatrix},$$

where we express an element $\sigma_j \in \mathcal{L}_j[t,\varepsilon]$, $\sigma_j = f_j + t\delta_t f_j + \varepsilon\delta_\varepsilon f_j$ as a column vector $(f_j, \delta_t f_j, \delta_\varepsilon f_j)^t$, $\delta_t g_{jk} \in \mathcal{O}_{jk,0}$, $\delta_\varepsilon g_{jk} \in \mathcal{O}_{jk,1}$ and g_{jk} is the transition function of $\mathcal{L}$. The cocycle condition for G_{jk} implies that $\{g_{jk}^{-1}(\delta_t g_{jk} + \delta_\varepsilon g_{jk})\}_{jk}$ is a 1-cocycle c_1 on $\mathcal{C}$ with values in $\mathcal{O}_{\mathcal{C}}$. Clearly, if we change c_1 by a coboundary, we get an isomorphic infinitesimal deformation of the invertible sheaf $\mathcal{L}$. Hence, the set of isomorphism classes of infinitesimal deformations of $\mathcal{L}$ is isomorphic to $H^1(\mathcal{C},\mathcal{O}_{\mathcal{C}})$. If we have a deformation $\mathcal{L}_{\mathcal{X}}$ of $\mathcal{L}$ over $(\mathcal{X},x)$ and $v : \mathcal{E} \to \mathcal{X}$ is a "tangent vector" to $\mathcal{X}$ at x, then the pull-back of $\mathcal{L}_{\mathcal{X}}$ under $id_{\mathcal{C}} \times v$ is an infinitesimal deformation of $\mathcal{L}$ and corresponds by the above argument to a class $[c_1] \in H^1(\mathcal{C},\mathcal{O}_{\mathcal{C}})$. This defines the Kodaira–Spencer map $KS : T_x\mathcal{X} \to H^1(\mathcal{C},\mathcal{O}_{\mathcal{C}})$ of the deformation.

Now we consider the deformation $\sigma \in H^0(\mathcal{C} \times_{\mathrm{Spec}(\Lambda)} \mathcal{E}, \mathcal{L}_{\mathcal{E}})$ of $s \in H^0(\mathcal{C},\mathcal{L})$. Let us write the local expression of σ as above: $\sigma_j = f_j + t\delta_t f_j + \varepsilon\delta_\varepsilon f_j$, where f_j is the local function representing s. Then, the cocycle condition for σ to be a global section reads

$$g_{jk}^{-1}\delta_t f_j = \delta_t f_k + g_{jk}^{-1}\delta_t g_{jk} f_k, \qquad g_{jk}^{-1}\delta_\varepsilon f_j = \delta_\varepsilon f_k + g_{jk}^{-1}\delta_\varepsilon g_{jk} f_k. \tag{3.1}$$

The meaning of these two equations is the following (see, e.g., [15] and the Appendix of [5]): the triple $(\{U_j\}_j, \{g_{jk}^{-1}(\delta_t g_{jk} + \delta_\varepsilon g_{jk})\}_{jk}, \{\delta_t f_j + \delta_\varepsilon f_j\}_j)$ gives rise to a class $\gamma_1 \in \mathbb{H}^1_s(\mathcal{C},\mathfrak{C}^\cdot)$ of the hyper-cohomology of the complex

$$\mathfrak{C}^\cdot : \quad 0 \to \mathcal{O}_{\mathcal{C}} \xrightarrow{s\cdot} \mathcal{L} \to 0$$

of sheaves on $\mathcal{C}$. The set of isomorphism classes of infinitesimal deformations of $(\mathcal{L}, s)$ is isomorphic to $\mathbb{H}^1_s(\mathcal{C},\mathfrak{C}^\cdot)$, and a corresponding Kodaira–Spencer map can be defined for any deformation.

Our goal is to show that the similarity between Eq. (3.1) and the second equation in (2.1) is not only formal, i.e., we can interpret Eq. (3.1) as the differential equation associated with a Kodaira–Spencer deformation of the spectral super line bundle together with its meromorphic sections,

$$g_{jk}^{-1}\partial_t f_j = (\partial_t + g_{jk}^{-1}\partial_t g_{jk}) f_k, \qquad g_{jk}^{-1}\partial_\varepsilon f_j = (\partial_\varepsilon + g_{jk}^{-1}\partial_\varepsilon g_{jk}) f_k. \tag{3.2}$$

To achieve this, we have first of all to construct a suitable family $\mathcal{L}_{\mathcal{X}}$ of line bundles on a Λ-super-curve $\mathcal{C}$ and then to interpret the Faà di Bruno polynomials as local representatives of sections of $\mathcal{L}_{\mathcal{X}}$. These two steps will be taken in Sections 4 and 5.

4. The universal relative positive super divisor

From now on we assume that $\mathcal{C}$ is a smooth super curve over Λ. Since the points of the super universal Grassmannian associated with a solution of the Jacobian super KP hierarchy belong to the component of index 0|0 we must require $\mathcal{C}$ to be a generic SKP curve and $\mathcal{L}$ to have degree g equal to the genus of $\mathcal{C}$.

Definition 4.1 (SKP curve [1]). A Λ-super-curve $\mathcal{C} = (C, \mathcal{O}_{\mathcal{C}})$ is called an *SKP curve* if its split structure sheaf $\mathcal{O}_{\mathcal{C}}^{\mathrm{sp}} := \mathcal{O}_{\mathcal{C}} \otimes_{\Lambda} \Lambda/\mathfrak{m}$ is of the form $\mathcal{O}_{\mathcal{C}}^{\mathrm{rd}}|\mathcal{S}$, where $\mathfrak{m}$ is the maximal ideal of nilpotent elements of Λ, $\mathcal{S}$ is an invertible $\mathcal{O}_{\mathcal{C}}^{\mathrm{rd}}$-module (a "reduced" line bundle) of degree 0 and $\cdot|\cdot$ denotes a direct sum of free Λ-modules, with on the left an evenly generated summand and on the right an odd one. If $\mathcal{S} \neq \mathcal{O}_{\mathcal{C}}^{\mathrm{red}}$ then $\mathcal{C}$ is called a *generic* SKP curve.
Let $\tilde{\mathcal{C}}$ be the dual super curve of $\mathcal{C}$, whose Λ-points are the irreducible superdivisors of $\mathcal{C}$ (see Appendix A). Constructing a universal family of line bundles $\mathcal{L}_{\mathcal{X}}$ requires the construction of the super Picard scheme of $\mathcal{C}$ and the corresponding super Poincaré sheaf. However, we can avoid this difficult step, since it suffices to produce the universal super divisor $\Delta^{(g)}$ of degree g. In analogy to the commutative case, the central object we have to consider is the gth symmetric product $S_g\tilde{\mathcal{C}}$ of the dual super curve $\tilde{\mathcal{C}}$, since $\tilde{\mathcal{C}}$ parameterizes irreducible positive super divisors on $\mathcal{C}$. Our discussion will follow closely that of [3], the only novelty being that we have to work over Λ rather than $\mathbb{C}$.

Let $\mathcal{C}^g := \mathcal{C} \times_{\mathrm{Spec}(\Lambda)} \cdots \times_{\mathrm{Spec}(\Lambda)} \mathcal{C}$ be the g-fold fibred product of $\mathcal{C}$ with itself over $\mathrm{Spec}(\Lambda)$. The symmetric group Σ_g of degree g acts on $\mathcal{C}^g$ by

$$\begin{array}{rccc} \Sigma_g \ni \sigma: & C^g & \to & C^g, \\ & (x_1, \dots, x_g) & \mapsto & (x_{\sigma(1)}, \dots, x_{\sigma(g)}), \end{array}$$

and

$$\begin{array}{rccc} \sigma: & \mathcal{O}_{\mathcal{C}}^{\otimes_\Lambda g} & \to & \mathcal{O}_{\mathcal{C}}^{\otimes_\Lambda g}, \\ & f_1 \otimes_\Lambda \cdots \otimes_\Lambda f_g & \mapsto & \left(\prod_{\substack{j<k \\ \sigma(j)>\sigma(k)}} (-1)^{\bar{f}_{\sigma(j)}\bar{f}_{\sigma(k)}}\right) f_{\sigma(1)} \cdots f_{\sigma(g)}, \end{array} \tag{4.1}$$

where C is the reduced curve associated with $\mathcal{C}$. We define the gth symmetric product of $\mathcal{C}$ to be the ringed space

$$S_g\mathcal{C} := \left(\frac{C^g}{\Sigma_g}, (\mathcal{O}^{\otimes_\Lambda g})^{\Sigma_g}\right),$$

whose structure sheaf is the graded sheaf of invariants of $\mathcal{O}^{\otimes_\Lambda g}$. Notice that, since σ is an even map (i.e., it preserves degrees), the action above is the same as that in Eq. (1) of [3]. The form given above makes the proof of the following proposition quite immediate.

Proposition 4.1. *The super space $S_g\mathcal{C}$ is a supermanifold over* $\mathrm{Spec}(\Lambda)$ *of dimension $g|g$.*

Proof. It is well known that $S_gC := C^g/\Sigma_g$ is a smooth scheme, so we have to show that locally $\mathcal{O}_{S_g\mathcal{C}}$ is isomorphic to $\mathcal{O}_{S_gC} \otimes \Lambda[\theta_1, \dots, \theta_g]$. Obviously $\Lambda \subset \mathcal{O}_{S_g\mathcal{C}}$. By definition there exists an open covering $\{U_j\}_{j\in J}$ of $\mathcal{C}$ such that $\mathcal{O}_{\mathcal{C}}(U_j) \simeq \Lambda \otimes_{\mathbb{C}} \mathcal{O}_C(U_j)[\theta_j]$. Let $p : C^g \to S_gC$ be the natural projection of ordinary schemes. One has only to prove that if V is an open affine subscheme of S_gC such that $\mathcal{O}_{\mathcal{C}^g}(U)$ $(U = p^{-1}V)$ is isomorphic to $\Lambda \otimes_{\mathbb{C}} (\mathcal{O}_C(U)[\theta])^{\otimes_{\mathbb{C}} g}$, then $\mathcal{O}_{S_g\mathcal{C}}(V) \simeq \mathcal{O}_{S_gC}(V) \otimes \Lambda[\varsigma_1, \dots, \varsigma_g]$ for suitable odd coordinates $\varsigma_1, \dots, \varsigma_g$. Now, $\sigma \in \Sigma_g$ acts as the identity on the first factor Λ: in fact we have

$$\begin{aligned}\sigma(\lambda_1 f_1 \otimes \cdots \otimes \lambda_g f_g) &= \sigma\left(\prod_{j<k}(-1)^{\bar{f}_j\bar{\lambda}_k}(\lambda_1\cdots\lambda_g) f_1 \otimes \cdots \otimes f_g\right)\\ &= \left(\prod_{\substack{l<m\\ \sigma(l)>\sigma(m)}}(-1)^{(\bar{f}_{\sigma(l)}+\bar{\lambda}_{\sigma(l)})(\bar{f}_{\sigma(m)}+\bar{\lambda}_{\sigma(m)})}\right)\lambda_{\sigma(1)} f_{\sigma(1)} \otimes \cdots \otimes\\ &\quad \lambda_{\sigma(g)} f_{\sigma(g)} = \left(\prod_{\substack{l<m\\ \sigma(l)>\sigma(m)}}(-1)^{\bar{f}_{\sigma(l)}\bar{\lambda}_{\sigma(m)}+\bar{f}_{\sigma(m)}\bar{\lambda}_{\sigma(l)}+\bar{f}_{\sigma(l)}\bar{f}_{\sigma(m)}}\right)\\ &\quad \times\left(\prod_{j<k}(-1)^{\bar{f}_{\sigma(j)}\bar{\lambda}_{\sigma(k)}}\right)\times \lambda_1\cdots\lambda_g f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(g)}\\ &= \left(\prod_{\substack{l<m\\ \sigma(l)>\sigma(m)}}(-1)^{\bar{f}_{\sigma(l)}\bar{\lambda}_{\sigma(m)}+\bar{f}_{\sigma(m)}\bar{\lambda}_{\sigma(l)}}\right)\left(\prod_{j<k}(-1)^{\bar{f}_{\sigma(j)}\bar{\lambda}_{\sigma(k)}}\right)\\ &\quad \times\lambda_1\cdots\lambda_g\sigma(f_1 \otimes \cdots \otimes f_g)\end{aligned}$$

and since

$$\left(\prod_{\substack{l<m\\ \sigma(l)>\sigma(m)}}(-1)^{\bar{f}_{\sigma(l)}\bar{\lambda}_{\sigma(m)}+\bar{f}_{\sigma(m)}\bar{\lambda}_{\sigma(l)}}\right)\left(\prod_{j<k}(-1)^{\bar{f}_{\sigma(j)}\bar{\lambda}_{\sigma(k)}}\right) = \prod_{j<k}(-1)^{\bar{f}_j\bar{\lambda}_k},$$

we get $\sigma(\lambda_1\cdots\lambda_g f_1 \otimes \cdots \otimes f_g) = \lambda_1\cdots\lambda_g\sigma(f_1 \otimes \cdots \otimes f_g)$. Therefore, it remains only to apply Theorem 1 of [3]. If (z, θ) are graded local coordinates of $\mathcal{C}$ then a system of graded local coordinates for $S_g\mathcal{C}$ is given by $(s_1, \dots, s_g, \varsigma_1, \dots, \varsigma_g)$, where $(s_1, \dots, s_g)$ are the (even) symmetric functions of $z_j = 1 \otimes \cdots \otimes z \otimes \cdots \otimes 1$ (with z in the jth position), $1 \le j \le g$ and $(\varsigma_1, \dots, \varsigma_g)$ are the odd symmetric functions defined by $\varsigma_j := \sum_{k=1}^{g}\theta_k\tilde{s}_{j-1}^{(k)}$, where $\theta_k := 1 \otimes \cdots \otimes \theta \otimes \cdots \otimes 1$ and $\tilde{s}_j^{(k)}$ is the jth symmetric function of $z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_g$. □

To exploit this construction, we give the following definition.

Definition 4.2. Let $\mathcal{X} = (X, \mathcal{O}_{\mathcal{X}})$ be a super scheme over $\mathrm{Spec}(\Lambda)$. A positive relative super divisor of degree g of $\mathcal{C} \times_{\mathrm{Spec}(\Lambda)} \mathcal{X} \to \mathcal{X}$ is a closed sub–super-scheme $\mathcal{Z}$ of

$\mathcal{C} \times_{\mathrm{Spec}(\Lambda)} \mathcal{X}$ of codimension $1|0$, defined by a homogeneous locally principal ideal $\mathcal{J}$ of $\mathcal{O}_{\mathcal{C}\times_{\mathrm{Spec}(\Lambda)}\mathcal{X}}$, such that $\mathcal{O}_{\mathcal{Z}}$ is a locally free $\mathcal{O}_{\mathcal{X}}$-module of rank $g|0$ and its reduction (modulo nilpotents) Z is a positive relative divisor of degree g of $C \times X \to X$.
By definition, then $\mathcal{Z}$ is locally defined by an equation of type

$$f = z^g - (a_1 + \theta\alpha_1)z^{g-1} + \cdots + (-1)^g(a_g + \theta\alpha_g) = 0,$$

where f is the local generator of $\mathcal{J}$ and the a_j's (respectively, the α_j's) are even (respectively, odd) local functions on $\mathcal{X}$. Our aim is to show that the symmetric product $S_g\tilde{\mathcal{C}}$ is the parameter space for the universal relative super divisor of degree g, $\Delta^{(g)}$, of $\mathcal{C}$. The universal relative super divisor of degree 1 is simply the sub–super-scheme $\Delta^{(1)}$ of $\mathcal{C} \times_{\mathrm{Spec}(\Lambda)} \tilde{\mathcal{C}}$ locally defined by the equation

$$z \otimes_\Lambda 1 - 1 \otimes_\Lambda \tilde{z} - \theta \otimes_\Lambda \tilde{\rho} = 0,$$

which we will write more compactly as $z - \tilde{z} - \theta\tilde{\rho} = 0$, where (z, θ) are local coordinates of $\mathcal{C}$ and $(\tilde{z}, \tilde{\rho})$ are the "dual" coordinates given in Eq. (A.2). Consider now the natural projections

$$\begin{array}{rccc} \pi_j: & \mathcal{C} \times_{\mathrm{Spec}(\Lambda)} \tilde{\mathcal{C}}^g & \to & \mathcal{C} \times_{\mathrm{Spec}(\Lambda)} \tilde{\mathcal{C}}, \\ & (x, \tilde{x}_1, \ldots, \tilde{x}_g) & \mapsto & (x, \tilde{x}_j), \end{array}$$

and define $\tilde{\Delta}_j := \pi_j^{-1}(\Delta^{(1)})$, $\tilde{\Delta}^{(g)} := \tilde{\Delta}_1 + \cdots + \tilde{\Delta}_g$. Since the local equation of $\tilde{\Delta}^{(g)}$ is

$$\prod_{j=1}^{g}(z - \tilde{z}_j - \theta\tilde{\rho}_j) = z^g - (s_1 + \theta\varsigma_1)z^{g-1} + \cdots + (-1)^g(s_g + \theta\varsigma_g) = 0,$$

where the s_j's and the ς_k's are the symmetric functions of the $\tilde{z}_m$'s and $\tilde{\rho}_n$'s we introduced at the end of the proof of Proposition 4.1, the next lemma holds true.

Lemma 4.2. *There exists a unique positive relative super divisor $\Delta^{(g)}$ of degree g of $\mathcal{C} \times_{\mathrm{Spec}(\Lambda)} S_g\tilde{\mathcal{C}} \to S_g\tilde{\mathcal{C}}$ such that $\pi^*(\Delta^{(g)}) = \tilde{\Delta}^{(g)}$, where $\pi : \mathcal{C} \times_{\mathrm{Spec}(\Lambda)} \tilde{\mathcal{C}}^g \to \mathcal{C} \times_{\mathrm{Spec}(\Lambda)} S_g\tilde{\mathcal{C}}$ is the natural projection.*
The most important result we need is Theorem 6 of [3], whose proof extends to the present situation.

Theorem 4.3. *The pair $(S_g\tilde{\mathcal{C}}, \Delta^{(g)})$ represents the functor of relative positive super divisors of degree g of $\mathcal{C}$, i.e., the natural map*

$$\begin{array}{rccc} R: & \mathrm{Hom}(\mathcal{X}, S_g\tilde{\mathcal{C}}) & \to & \mathrm{Div}^g_{\mathcal{X}}(\mathcal{C} \times_{\mathrm{Spec}(\Lambda)} \mathcal{X}), \\ & f & \mapsto & (id \times f)^*\Delta^{(g)}, \end{array}$$

is a functorial isomorphism for every Λ-super-scheme $\mathcal{X}$.

5. The geometric super Faà di Bruno polynomials

The constructions of Section 4 allow us to define a canonical family of super line bundles together with an even section. For simplicity, we call $\mathcal{X} := S_g\tilde{\mathcal{C}}$ and we also assume that $\mathcal{C}$ is a generic SKP super curve of genus g. Then, $\mathcal{L}_\mathcal{X} := \mathcal{O}_{\mathcal{C}\times_{\mathrm{Spec}(\Lambda)}\mathcal{X}}(\Delta^{(g)})$ is an $\mathcal{X}$-family of super line bundles on $\mathcal{C}$ and $\Delta^{(g)}$ defines a section σ of $\mathcal{L}_\mathcal{X}$. If we let $\mathcal{L}$ be any non-special super line bundle on $\mathcal{C}$ (i.e., such that the reduced invertible sheaf $\mathcal{L}^{\mathrm{rd}}$ is non-special on C) of degree g and we call $s_\mathcal{L}$ the unique (up to multiplication by a complex number) section that generates the even part of $H^0(\mathcal{C},\mathcal{L})$, then the divisor $(s_\mathcal{L})$ can be thought of as a $\mathrm{Spec}(\Lambda)$-family of positive relative super divisors of degree g and the universality property of $\Delta^{(g)}$ (Theorem 4.3) gives a unique map $f_\mathcal{L} : \mathrm{Spec}(\Lambda) \to \mathcal{X}$, i.e., a Λ-point x of $\mathcal{X}$, such that $(s_\mathcal{L}) = (id \times f_\mathcal{L})^*\Delta^{(g)}$. In turn, this induces an isomorphism $\rho_\mathcal{L} : \mathcal{L} \to (id \times f_\mathcal{L})^*\mathcal{L}_\mathcal{X}$ such that $(id \times f_\mathcal{L})^*\sigma = \rho_\mathcal{L} s_\mathcal{L}$, so we can interpret the triple $(\mathcal{X}, \mathcal{L}_\mathcal{X}, \sigma)$ as a deformation of $(\mathcal{L}, s_\mathcal{L})$ for any non-special super line bundle $\mathcal{L}$ on $\mathcal{C}$. Finally, if we put graded super coordinates $\mathbf{t} = (t_1, \ldots, t_{2g})$ on $\mathcal{X}$ ($\bar{t}_j = j \bmod 2$) then the cocycle conditions (3.1) for the section σ as a deformation of $s_{\mathbf{t}_0} := \sigma|_{\mathcal{C}\times\{x(\mathbf{t}_0)\}}$, for any $\mathbf{t}_0$, become the differential equations

$$g_{jk}^{-1}\partial_{t_l} f_j = \partial_{t_l} f_k + (g_{jk}^{-1}\partial_{t_l} g_{jk}) f_k,$$

which are manifestly of the form of (2.1). To accomplish our goal of describing the algebraic geometric super Faà di Bruno polynomials, we have therefore only to choose a suitable open covering of $\mathcal{C} \times_{\mathrm{Spec}(\Lambda)} \mathcal{X}$ and to appropriately select two coordinates t_{2j} and t_{2k+1} and to call them x and φ, respectively.

As before, select a non-special super line bundle $\mathcal{L}$ of degree g on $\mathcal{C}$. Let $p_\infty \in C$ be a reduced point of C such that it is not Weierstrass for C and the reduced section $s_\mathcal{L}^{\mathrm{rd}}$ does not vanish at p_∞. Let $U_0 \subset C$ be an open neighbourhood of p_∞ where we can define graded coordinates (z,θ) for $\mathcal{C}$ centred at p_∞ (i.e., $z(p_\infty) = 0$) and let $U_1 := C - \{p_\infty\}$. Then, $\{U_0, U_1\}$ is a Stein open covering of $\mathcal{C}$. Since $s_\mathcal{L}^{\mathrm{rd}}$ does not vanish at p_∞, the section $s_\mathcal{L}$ gives a local trivialization η of $\mathcal{L}$ on U_0 (suitably restricted). Then, the quintuple $(\mathcal{C}, D := (z)|_{U_0}, (z,\theta), \mathcal{L}, \eta)$ defines through the super Krichever map a point of SGr_Λ. Finally, let $\mathcal{V}$ be a Stein open neighbourhood of $(s_\mathcal{L}) \in \mathcal{X}$ where the coordinates $\mathbf{t}$ are defined. The open subsets $\mathcal{U}_0 := U_0 \times_{\mathrm{Spec}(\Lambda)} \mathcal{V}$ and $\mathcal{U}_1 := U_1 \times_{\mathrm{Spec}(\Lambda)} \mathcal{V}$ define a Stein covering of $\mathcal{C} \times_{\mathrm{Spec}(\Lambda)} \mathcal{V}$ over which we can trivialize $\mathcal{L}_\mathcal{V} := \mathcal{L}_\mathcal{X}|_{\mathcal{C}\times_{\mathrm{Spec}(\Lambda)}\mathcal{V}}$. Restricting $\mathcal{V}$ if necessary, we can assume that σ gives a local trivialization of $\mathcal{L}_\mathcal{V}$ over $\mathcal{U}_0$.

Now, we move to the analytic category instead of the algebraic one. Let $\mathcal{N} := \pi^*\mathcal{O}_\mathcal{C}(gD)$, where π is now the projection of $\mathcal{C} \times_{\mathrm{Spec}(\Lambda)} \mathcal{V}$ to $\mathcal{C}$, and let μ be the pull back by π of the section of $\mathcal{O}_\mathcal{C}(gD)$ which generates the even part of its module of global sections. Then, μ gives a local trivialization of $\mathcal{N}$ over $\mathcal{U}_1$. Since $\mathcal{L}_\mathcal{V} \otimes \mathcal{N}^{-1}$ has relative degree 0 it follows that restricting again $\mathcal{V}$ if necessary, it has a local analytic trivialization ν over $\mathcal{U}_1$ and $\tau = \nu\mu$ gives a trivialization of $\mathcal{L}_\mathcal{V}$ over U_1.

Summarizing, we have a trivialization (σ, τ) of $\mathcal{L}_\mathcal{V}$ over $(\mathcal{U}_0, \mathcal{U}_1)$, with respect to which σ is represented by the couple of functions $(f_0 = 1, f_1)$ and the transition function of $\mathcal{L}_\mathcal{V}$

is $g_{10} = f_1/f_0$. Let us define $\hat{H}^{(k)} := \partial_{t_k} \log g_{10}$. Then, these meromorphic functions on U_0 satisfy Eq. (2.4) and are therefore our candidates for the super currents of the hierarchy.

Observe that it is possible to choose the coordinates t_k in such a way that (multiplying τ by the exponential of a suitable meromorphic function whose poles are only over $\pi^{-1}(p_\infty)$) $\hat{H}^{(h)}$ has the correct asymptotic behaviour (2.2) (here our coordinate z is the inverse of the z appearing there). Notice also that $\hat{H}^{(k)}|_{\mathcal{U}_0 \cap \mathcal{U}_1}$ represents the class of $H^1(\mathcal{C} \times_{\mathrm{Spec}(\Lambda)} \mathcal{V}, \mathcal{O}_{\mathcal{C} \times_{\mathrm{Spec}(\Lambda)} \mathcal{V}})$ corresponding to the deformation of $\mathcal{L}$ along t_k. Since the asymptotic behaviour of $\hat{H}^{(1)}$ is $\theta + \mathrm{O}(z)$ it follows that the first time t_1 does not deform $\mathcal{L}$ at all. The super Jacobian $\mathrm{Jac}(\mathcal{C})$ of $\mathcal{C}$ has dimension $g|g-1$, while $S_g\tilde{\mathcal{C}}$ has dimension $g|g$ and maps surjectively to $\mathrm{Pic}^g(\mathcal{C}) \simeq \mathrm{Jac}(\mathcal{C})$, hence there is an odd direction in $\mathcal{V}$ which corresponds to trivial deformations of $\mathcal{L}$, i.e., there exists indeed a coordinate like t_1.

In Section 2.3 of [6], we have shown that the Faà di Bruno generator is computed by the formula $\hat{h} := \hat{H}^{(1)}|_{t_1+\varphi} + \varphi \hat{H}^{(2)}|_{t_1+\varphi}$. The cocycle condition (3.1) can be interpreted also as saying that $(\partial_{t_k} f_0 + \hat{H}^{(k)} f_0, \partial_{t_k} f_1)$ is a section of $\mathcal{L}_{\mathcal{V}}(\infty\pi^* D)$ with pole of order k at $\pi^* D$. Thus, the super Faà di Bruno recurrence relation (2.1) corresponds to deformation along the non-integrable vector field $\mathcal{D}$ and the super Faà di Bruno polynomials $\hat{h}^{(k)}$ are the local representatives on $\mathcal{U}_0$ of the meromorphic sections $\sigma^{(k)}$ of $\mathcal{L}_{\mathcal{V}}$ obtained by iterative deformation of $\sigma^{(0)} := \sigma$ along $\mathcal{D}$. The form of $\hat{h}$ implies also that the $\sigma^{(k)}$'s form a basis of $H^0(\mathcal{C} \times_{\mathrm{Spec}(\Lambda)} \mathcal{V}, \mathcal{L}_{\mathcal{V}}(\infty\pi^* D))$ over $\mathcal{O}_{\mathcal{V}}$. We can restate the above discussion in the following proposition.

Proposition 5.1. *The super Faà di Bruno recurrence relation is the cocycle condition for the hypercohomology group describing the deformations of the dynamical super line bundle $\mathcal{L}$ on the spectral curve $\mathcal{C}$ and of its meromorphic sections which give rise to the super Krichever map.*

We end by remarking that Eq. (2.3) is an obvious consequence of the definition of $\hat{h}$ and $\hat{H}^{(k)}$.

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Appendix A. Super curves

In this appendix we recall some facts concerning super curves, referring to [7] for more details on supergeometry.

Let Λ be a Grassmann algebra over $\mathbb{C}$. An algebraic super curve over Λ, also called a *Λ-super-curve* for brevity, is a proper irreducible superscheme $\mathcal{C} \to \mathrm{Spec}(\Lambda)$ over $\mathrm{Spec}(\Lambda)$ with fibre dimension 1|1 and whose underlying reduced scheme is a proper irreducible algebraic curve over $\mathbb{C}$. Throughout this paper we assume $\mathcal{C}$ to be a supermanifold, so it is given by a pair $(C, \mathcal{O}_{\mathcal{C}})$, where C is a topological space and $\mathcal{O}_{\mathcal{C}} = \mathcal{O}_{\mathcal{C},0} \oplus \mathcal{O}_{\mathcal{C},1}$ is a sheaf of super-commutative Λ-algebras on C such that

1. $(C, \mathcal{O}_C := \mathcal{O}_{\mathcal{C}}^{\mathrm{rd}} = \mathcal{O}_{\mathcal{C}}/\mathcal{J}_{\mathcal{C}})$ is a smooth irreducible proper algebraic curve over $\mathbb{C}$, where $\mathbb{J}_{\mathcal{C}}$ is the ideal sheaf $\mathcal{O}_{\mathcal{C},1} + \mathcal{O}_{\mathcal{C},1}^2$,
2. there exists an open covering $\{U_j\}_{j\in J}$ of C and odd elements $\theta_j \in \mathcal{O}_{\mathcal{C}}(U_j)$ such that

$$\mathcal{O}_{\mathcal{C}}(U_j) \simeq \mathcal{O}_C(U_j) \otimes_{\mathbb{C}} \Lambda[\theta_j].$$

A Λ-point of $\mathcal{C}$ is a map $\mathrm{Spec}(\Lambda) \to \mathcal{C}$ whose composition with the projection $\mathcal{C} \to \mathrm{Spec}(\Lambda)$ is the identity morphism.

An invertible sheaf $\mathcal{L}$ on $\mathcal{C}$ is a locally free evenly generated $\mathcal{O}_{\mathcal{C}}$-module of rank 1|0; it is the sheaf of sections of a super line bundle that, abusing notations, we still call $\mathcal{L}$. We can find a suitable open covering $\{U_j\}_{j\in J}$ of $\mathcal{C}$ over the elements of which $\mathcal{L}$ is trivial. Then the super line bundle is completely described in terms of its (even invertible) transition functions $g_{jk} \in \Gamma(U_j \cap U_k, \mathcal{O}_{\mathcal{C},0}^{\times})$ satisfying the usual cocycle conditions. The set of isomorphism classes of super line bundles on $\mathcal{C}$ is therefore $H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C},0}^{\times})$ and tensor product of invertible sheaves (or, equivalently, multiplication in $\mathcal{O}_{\mathcal{C},0}^{\times}$) gives it a group structure under which it is called the Picard group $\mathrm{Pic}(\mathcal{C})$ of $\mathcal{C}$.

Another way to describe an invertible sheaf is by means of super (Cartier) divisors. A super divisor on $\mathcal{C}$ is a collection $D := \{(U_j, f_j)\}_{j\in J}$ of even non-zero rational functions f_j defined, up to even invertible regular functions, on the open subsets U_j of a covering of $\mathcal{C}$, and agreeing in the intersections $U_j \cap U_k$ up to an element of $\mathcal{O}_{\mathcal{C},0}^{\times}(U_j \cap U_k)$, i.e., D is a section of $Rat_{\mathcal{C},0}^{\times}/\mathcal{O}_{\mathcal{C},0}^{\times}$, where $Rat_{\mathcal{C}}$ is the sheaf of rational functions on $\mathcal{C}$. With the super divisor D one associates the invertible subsheaf $\mathcal{O}_{\mathcal{C}}(D) \subset Rat_{\mathcal{C}}$ whose local sections over U_j span the module $f_j^{-1}\mathcal{O}_{\mathcal{C}}(U_j)$ and the transition functions of the corresponding super line bundle are $g_{jk} = f_j f_k^{-1}$. We have the exact sequence

$$0 \to \mathcal{O}_{\mathcal{C},0}^{\times} \to Rat_{\mathcal{C},0}^{\times} \to Rat_{\mathcal{C},0}^{\times}/\mathcal{O}_{\mathcal{C},0}^{\times} \to 0$$

and a super divisor D is called principal if it is the image of a global non-zero even rational function f, in which case we write $D = (f)$. Of course, the invertible sheaf associated with a principal divisor is trivial and vice versa. D is called effective (or positive) if f_j is regular for every j, and irreducible if $f_j = z_j - \tilde{z}_j - \theta_j\tilde{\theta}_j$, where $\tilde{z}_j, \tilde{\theta}_j \in \Lambda$.

A useful concept associated with irreducible super divisors is the dual super curve $\tilde{\mathcal{C}}$ of $\mathcal{C}$, which we briefly review (see [1,2] for more details). Let $\underline{\mathcal{C}} = (C, \mathcal{O}_{\underline{\mathcal{C}}})$ be the $N = 2$ super curve whose reduced algebraic curve is again C and whose structure sheaf is the only super conformal extension of $Ber_{\mathcal{C}}$ by $\mathcal{O}_{\mathcal{C}}$

$$0 \to \mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\underline{\mathcal{C}}} \to Ber_{\mathcal{C}} \to 0. \tag{A.1}$$

Here $Ber_{\mathcal{C}}$ is the dualizing sheaf of $\mathcal{C}$, whose transition functions g_{jk} are the Berezinians of the (super) Jacobian matrices of the coordinate transformations between U_j and U_k, and the super conformal property means that the local form $\omega_j := \mathrm{d}z_j - \mathrm{d}\theta_j\rho_j$ is globally defined up to a scalar factor, where (z_j, θ_j, ρ_j) are graded local coordinates on $\underline{\mathcal{C}}$ adapted to $\mathcal{C}$ (i.e., (z_j, θ_j) are coordinates on $\mathcal{C}$). The kernel of ω_j is generated by $\mathcal{D}_j := \partial_{\rho_j}$ and $\tilde{\mathcal{D}}_j := \partial_{\theta_j} + \rho_j\partial_{z_j}$ and one can easily convince himself that $\mathcal{D}_j$ represents locally the map $\mathcal{O}_{\underline{\mathcal{C}}} \to Ber_{\mathcal{C}}$, thus the structure sheaf $\mathcal{O}_{\mathcal{C}}$ of $\mathcal{C}$ is the kernel of $\mathcal{D}$.

Introducing the new coordinates

$$\tilde{z}_j := z_j - \theta_j \rho_j, \qquad \tilde{\theta}_j := \theta_j, \qquad \tilde{\rho}_j := \rho_j \tag{A.2}$$

on $\underline{\mathcal{C}}$, the two operators above become $\mathcal{D}_j = \partial_{\tilde{\rho}_j} + \tilde{\theta}_j \partial_{\tilde{z}_j}$ and $\tilde{\mathcal{D}}_j = \partial_{\tilde{\theta}_j}$, respectively, so the kernel of $\tilde{\mathcal{D}}_j$ consists of functions of $\tilde{z}_j$ and $\tilde{\rho}_j$. One shows that this makes sense globally obtaining therefore a new exact sequence

$$0 \to \mathcal{O}_{\tilde{\mathcal{C}}} \to \mathcal{O}_{\underline{\mathcal{C}}} \stackrel{\tilde{\mathcal{D}}}{\to} \mathcal{Q} \to 0,$$

where $\mathcal{O}_{\tilde{\mathcal{C}}}$ is the structure sheaf of a $1|1\Lambda$-super-curve $\tilde{\mathcal{C}}$ which is called the dual super curve of $\mathcal{C}$, moreover $\mathcal{Q} \simeq Ber_{\tilde{\mathcal{C}}}$ and $\tilde{\tilde{\mathcal{C}}} \simeq \mathcal{C}$, which explains the terminology. The interesting fact is that the Λ-points of $\tilde{\mathcal{C}}$ correspond to the irreducible divisors of $\mathcal{C}$.

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